A PARTICULAR CASE OF THE MOTION OF A RIGID BODY ABOUT A FIXED POINT IN A NEWTONIAN FORCE FIELD

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PMM Vol.27, No.1, 1963, pp. 175-178

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(Received November 20, 1962)

We consider the following motion of a body about a fixed point in a Newtonian force field: We assume that the body has dynamic symmetry ($A = B$), that the fixed point coincides with the center of mass of the body, and that the initial conditions are the following: the transverse components on angular velocity are equal to zero ($p_0 = q_0 = 0$) and the longitudinal component is arbitrary ($r_0 \neq 0$). In this case, as is known, the axis of symmetry of the body would maintain a fixed direction if it were not subjected to the Newtonian force field. It is therefore the existence of a Newtonian force field that produces all the effects and motions of the axis of the body in the present example. The action of the Newtonian field enters into this example in its pure form, uncomplicated by more general initial conditions. This is the fact that makes this type of motion worth studying.

This problem of a motion of a rigid body about a fixed point can be integrated by quadratures as a particular case of two more general integrable cases [1-3].

Let θ, ψ, φ be the Eulerian angles, where θ is the angle between the axis of symmetry of the body and the line from the center of attraction to the motionless center of mass of the body. The integrals of the energy and angular momentum then become

$$\sin^2\theta\dot{\psi}^2 + \dot{\theta}^2 = \alpha + m\omega^2\cos^2\theta$$
$$\sin^2\theta\dot{\psi} = \beta - br_0\cos\theta \qquad \left(\omega^2 = \frac{\mu}{R^3}, \quad m = 3\frac{A-C}{A}, \quad b = \frac{C}{A}\right)$$

Here r_0 is the component of angular velocity along the axis of symmetry (r_0 remains constant by virtue of the equations of motion); α and β are constants of integration; C is the longitudinal moment of inertia; A is the transverse moment of inertia; μ is the gravitational constant; R is the distance from the center of mass of the body to the center of attraction.

The variables in this system are easily separable, and the problem reduces to quadratures

$$\left(\frac{du}{dt}\right)^2 = (1-u^2)\,[\alpha + m\omega^2 u^2] - [\beta - br_0 u]^2 \equiv f(u), \qquad \frac{d\psi}{dt} = \frac{\beta - br_0 u}{1-u^2}, \qquad u = \cos\theta$$

For our initial conditions ($p_0 = q_0 = 0$) we have

$$\alpha = -m\omega^2 u_0^2, \qquad \beta = br_0 u_0$$

Consequently

$$\left(\frac{du}{dt}\right)^2 = f(u), \quad f(u) = (u_0 - u)\,\{-(1-u^2)(u+u_0)\,m\omega^2 - (u_0 - u)\,b^2 r_0^2\} \tag{1}$$

$$\frac{d\psi}{dt} = br_0 \frac{u_0 - u}{1-u^2} \tag{2}$$

The motion can be analyzed without inverting the elliptic quadratures (1) and (2). It follows from (1) that the motion takes place in an interval of values of u bounded by two values: the initial value u_0, and a value u_1, related to u_0 by the formula

$$u_0 = u_1 \frac{b^2 r_0^2 - (1-u_1)^2 m\omega^2}{b^2 r_0^2 + (1-u_1^2)\,m\omega^2} \tag{3}$$

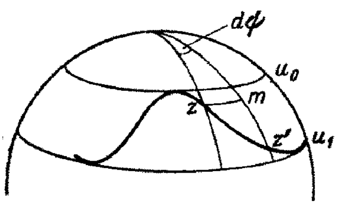

Fig. 1.

On a unit sphere with center at the center of mass of the body, the trace of the axis of the body will thus describe a curve (Fig. 1) between the parallels u_0 and u_1.

Let V be the angle between the curve described by the trace of the axis of the body and a meridian on the unit sphere; $zm = \sin\theta\, d\psi$, $z'm = d\theta$, as can be seen from Fig. 1.

Then

$$\tan V = -(1-u^2)\frac{d\psi}{du} = -br_0\sqrt{1-u^2}\,\frac{u-u_0}{\sqrt{f(u)}}$$

It follows from this that $\tan V = 0$ if $u = u_0$, that is, there is a reversal point on the parallel $u = u_0$; $\tan V = \infty$ if $u = u_1$, that is, the trajectory is tangent to the parallel u_1.

On the parallel u_0 the precession velocity (2) is zero: $\dot\psi(u_0) = 0$;

on the parallel u_1 the precession velocity, as can easily be shown, reaches its maximum value $\dot{\psi}(u_1) = \dot{\psi}_{max}$. We introduce the average precession velocity

$$\langle\dot{\psi}\rangle = \frac{1}{2}\,[\dot{\psi}(u_0) + \dot{\psi}(u_1)] = \frac{1}{2}\,\dot{\psi}(u_1) \tag{4}$$

Substituting (3) into (2) and taking (4) into account, we find

$$\langle\dot{\psi}\rangle = -3\omega^2 \frac{A-C}{Cr_0}\,\frac{\cos\theta_1}{1+3\sin^2\theta_1\,(\omega/r_0)^2\,[(A-C)/C]\,(A/C)} \tag{5}$$

Here θ_1 is determined by θ_0 and the parameters of the problem, in accordance with (3). If $\omega/r_0 \ll 1$, that is, the effect of the perturbations is small, then, to within terms of second order, we have

$$\langle\dot{\psi}\rangle \approx -3\omega^2 \frac{A-C}{Cr_0}\cos\theta_0 \tag{6}$$

Let us consider formula (3). The curves for various values of the parameter

$$\zeta = b^2 r_0{}^2/m\omega^2 \tag{7}$$

are schematically represented in Fig. 2. From these curves we can understand the nature of the motion. Let us first consider the region $\zeta > 0$, or, stated in another way, $m > 0$, that is, when the body is elongated. This region is included between the diagonals $u_0 = \pm\, u_1$ and contains the u_1-axis. If $\zeta = \zeta_2 > 1$, then each value of u_0 is associated with only one value of u_1, which is closer to u_0 for larger values of ζ; $|u_1| > |u_0|$, and u_1 and u_0 have the same sign.

The curve for $\zeta = 1$ is tangent to the u_1-axis. For $\zeta = \zeta_1 < 1$ each value of u_0 can be associated with only one or three values of u_1 (the polynomial $f(u)$ will have two or four real roots including u_0). But in the

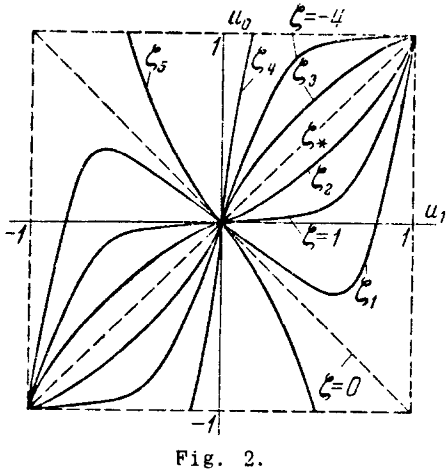

Fig. 2.

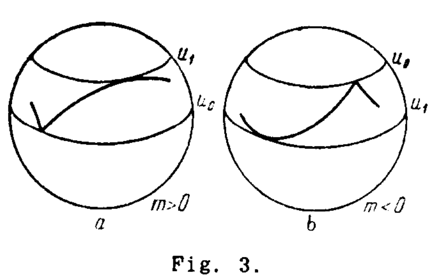

Fig. 3.

case of a real motion, the sign of u_1 must be the same as that of u_0

(this is easily seen if we consider the function $f(u)$ for this case, in accordance with formula (1)). We will again have the condition $|u_1| > |u_0|$.

In the case under consideration ($\zeta > 0$) the motion will have the following nature: The parallel u_1 lies closer to the pole of the unit sphere than the parallel u_0. The curve, as was indicated earlier, is tangent to the parallel u_1 and has a point of reversal at u_0 (Fig. 3*a*). For the case of a compressed body $m < 0$, that is, $\zeta < 0$. In this case for each value of u_0 we will always have one value of u_1 (Fig. 2). Moreover, if $\zeta < -1$ (for example, $\zeta = \zeta_3$ or $\zeta = \zeta_4$ in Fig. 2), then u_1 will have the same sign as u_0, and $|u_1| < |u_0|$. The points of reversal will lie on the initial parallel u_0, which is closer to the pole than is u_1 (Fig. 3*b*). If, on the other hand, $\zeta = \zeta_5 > -1$, then u_1 and u_0 will have opposite signs, and the trace of the axis of the satellite as it moves in its path will intersect the equator of the unit sphere; qualitatively the motion will be of the type shown in Fig. 3*b*.

It should be noted that the curve $\zeta = -4$ is tangent (Fig. 2) to the horizontal lines $u_0 = \pm 1$ at the points $u_1 = \pm 1$. It follows from this that the motion $u_0 = u_1 = \pm 1$ (rotation about an axis which coincides with the direction toward the center of attraction) will be stable if $|\zeta| = |\zeta_3| > 4$ and will be unstable if $|\zeta| = |\zeta_4| < 4$, since in the first case an infinitesimal deviation of u_0 from unity will correspond to an infinitesimal deviation of u_1 from unity, whereas in the second case, as can be seen from Fig. 2, no matter how small u_0 deviates from unity, u_1 will deviate from unity by a finite amount. If $\zeta > 0$, that is, if the body is elongated, then, as can be seen from Fig. 2, the motion $u_0 = u_1 = \pm 1$ is always stable. Thus, the necessary and sufficient conditions for stability of rotation about a vertically oriented axis will be the conditions

$$\zeta > 0, \qquad \zeta < -4$$

This result agrees with one of the results obtained in [2], where the stability conditions were obtained by the Liapunov-Chetaev method.

We observe that $u_1 \to \pm 1$ as $\zeta \to +0$, which corresponds to a conversion of spatial motion into plane motion as $r_0 \to 0$. In this case $\langle\dot{\psi}\rangle \to 0$, as can be seen from (5), and the trace of the axis of symmetry of the elongated body will oscillate, passing through the pose ($u_1 = +1$ or -1, depending on the sign of u_0), between the extreme values bounded by the parallel $\theta_0 = \arccos u_0$.

If $\zeta \to -0$, then $u_1 \to -u_0$, and the axis of symmetry of the compressed body will oscillate, passing through the equator, between parallels which are equidistant from the equator. The limiting line

$\zeta = \zeta_* = \pm\infty$ is also shown in Fig. 2.

If r_0 is close to zero, or, in any event, $r_0 \ll \omega$, then (for the case of a compressed body)

$$\langle\dot{\psi}\rangle \approx \frac{C}{A} r_0 \frac{\cos\theta_0}{\sin^2\theta_0}$$

and the period of precession will be

$$T_\psi = \frac{2\pi}{r_0}\frac{A}{C}\frac{\sin^2\theta_0}{\cos\theta_0}$$

The period of nutation will be close to the period of plane oscillations [1]

$$T_\theta = \frac{4K(k^2)}{\sqrt{3\omega^2(C-A)/C}}, \qquad k^2 = \cos^2\theta_0$$

Here K is the complete elliptic integral of the first kind. Let us consider the ratio of periods

$$\frac{T_\theta}{T_\psi} = \frac{2K(\cos^2\theta_0)}{\pi\sqrt{3(C-A)/C}}\frac{C}{A}\frac{r_0}{\omega}\frac{\cos\theta_0}{\sin^2\theta_0} \tag{8}$$

Since

$$\frac{2}{\pi}K(\cos^2\theta_0) = 1 + \left(\frac{1}{2}\right)^2\cos^2\theta_0 + \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\cos^4\theta_0 + \dots$$

it follows that for the case of small oscillations in a narrow band about the equator ($\theta_0 = 90^\circ - \theta^*$, where θ^* is small) we will have approximately

$$\frac{T_\theta}{T_\psi} \approx \frac{C/A\sin\theta^*}{\sqrt{3(C-A)/C}}\frac{r_0}{\omega}$$

Since, by assumption, $r_0 \ll \omega$ and θ^* is small, it follows that during one precession period there will be a great many nutation periods, that is, a small segment of the trajectory on the unit sphere will contain many "petals". However, as the distance of the initial point from the equator increases, there will be an increase not only in the amplitude of the nutation oscillations, but also in the width of the trajectory petals (since $K(\cos^2\theta_0)$ in the numerator of formula (8) increases while $\sin^2\theta_0$ in the denominator decreases). It should also be noted that in the general case the difference $\Delta = u_1 - u_0$ between the cosines of the boundary latitudes is given by the formula

$$\Delta = 2\cos\theta_1\sin^2\theta_1\frac{\varkappa}{1+\varkappa\sin^2\theta_1} \qquad \left(\varkappa = \frac{1}{\zeta}\right) \tag{9}$$

This follows from (3).

For the case of very rapid rotation of the body, we have $\omega/r_0 \ll 1$, $\kappa \ll 1$, and it follows from Fig. 2 that $\theta_1 = \theta_0 + \delta$, where δ is small. Then, by formula (3)

$$\Delta = 2\kappa u_0 (1 - u_0^2), \quad \text{or} \quad \Delta = 6\left(\frac{\omega}{r_0}\right)^2 \left(\frac{A-C}{C}\right) \frac{A}{C} \cos\theta_0 \sin^2\theta_0 \tag{10}$$

In this case, in accordance with (6), the period of precession will have a large value, of the order of r_0/ω

$$T_\psi = \frac{2\pi}{3\,[(A-C)\,/\,C]\cos\theta_0\omega}\left(\frac{r_0}{\omega}\right)$$

Let us estimate the period of the nutation oscillations for a rapidly rotating body.

Let $u = u_0 + x$, where x is small. Then, by (1), leaving only first-order and second-order terms under the radical sign, we have

$$\frac{dx}{dt} = \sqrt{x\,\{2m\omega^2 u_0 (1 - u_0^2) - x b^2 r_0^2\}}$$

Hence the half-period of the nutation oscillations is

$$\frac{\tau}{2} = \int_0^\Delta \frac{dx}{\sqrt{x\,\{2m\omega^2 u_0 (1 - u_0^2) - x b^2 r_0^2\}}} = \frac{\pi}{b r_0}$$

We see that the oscillations in the nutation angle will be rapid. The trajectory on the unit sphere will contain a large number of fine petals.

In conclusion, we observe that if $A \gg C$, the motion will approach the form of plane oscillations (in the same sense as for $r_0 \to 0$), since in the limiting case ($C = 0$) formulas (1) and (2) immediately yield plane oscillations.

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Translated by A.S.